



Kant's Antinomies of Pure Reason and the 'Hexagon of Predicate Negation'

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Abstract. Based on an analysis of the category of “infinite judgments” in Kant, we will introduce the logical hexagon of predicate negation. This hexagon allows us to visualize in a single diagram the general structure of both Kant’s solution of the antinomies of pure reason and his argument in favor of Transcendental Idealism.

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1. Introduction

Negation is an essential feature of communication and thought; and natural languages offer various ways for negating assertions and for making negative claims. We can reject the proposition “S is P” in various ways. We can say “No! It is not true that S is P” or “No! S is not P” or “No! S is non-P”, depending on whether the negation is applied to the whole proposition, the copula, or the predicate.

All of these statements involving S and P contain a negative particle. Standard formal logic uses only one kind of negation, understood as an operator which is applied to whole propositions: “not (S is P)”. In natural language, for which sentential negation can be ambiguous ([7], 364), the unambiguous sentential negation is most closely approximated by sentences like “It is not true/it is not the case/it is false, that S is P”. However, such examples contain semantic terms like ‘true’ and ‘false’ which are foreign to the object language of formal logic.

Are the three kinds of negative propositions synonymous and does the negation operator from propositional logic suffice for all of them? This question does not seem to have excited the minds during the history of formal logic. Gottlob Frege [6], arguing against the existence of negative propositions *sui generis*, put modern formal logic on its current track from the beginning. Rare were those who, like Fred Sommers [19], rebelled against the “Fregean dogma”

TABLE 1. Three kinds of negations

1	not (S is P)	non (S est P)	Negation of the proposition
2	S isn't P	S non-est P	Predicate denial or negation of the copula
3	S is non-P	S est non-P	Affixal negation or negation of the predicate

Anticipating a device used by Kant to distinguish two kinds of negation, we include a Latin version, which allows us to distinguish between negation of the copula and negation of the predicate by the word order: *non est* and *est non*

and defended the view that form 2 (in Table 1) is not reducible to form 1. Form 3, the use of negative predicates, attracted even less attention. Frege dismissed it without further ado on the grounds that the question of how to deal with affixal negation, i.e. negative/negated predicates of the type ‘un-P’, ‘in-P’, or ‘P-less’ like *unintelligible*, *infinite*, *immortal*, *fearless*, and so on belongs to semantics, not logic [6].

Immanuel Kant, to a certain extent, had opinions very much like Frege’s and in the *Critique of Pure Reason* excluded the distinction between negation of the copula and negation of the predicate from “general” logic, which “abstracts from all content [*Inhalt*] of the predicate (even if it is negative), and considers only whether it is attributed to the subject or opposed to it” (B97).¹ However, Kant went on to introduce precisely this distinction within the framework of *transcendental* logic. In the presentation of the Table of Judgments (B95–B97) he envisions three “qualities” of judgment: *affirmative*, *negative* and *infinite* (Fig. 1). Affirmative judgments have the form ‘S is P’ (or ‘S *est* P’). Negative judgments have the form ‘S is not P’ (or ‘S *non est* P’). And infinite judgments have the form: ‘S is non-P’ (or ‘S *est non* P’). Although infinite judgments (propositions with negative predicates) do not belong to logic proper, they do play a crucial role in two of the three chapters of the *Dialectics*: the “Antinomies” and the “Ideal.”² As we will argue below, each of the antinomies, can be formulated as an opposition between an affirmative judgment and an infinite judgment about the same subject: ‘S is P’ vs. ‘S is non-P’. The resolution will involve the negative judgment ‘S is not P’ (or ‘S is neither P nor non-P’). With regard to the “mathematical” antinomies (size of the world and division of matter) Kant actually exploits this form of opposition in the presentation of the antinomy.

¹Kant reiterates this position in his *Lectures on Logic*: “since logic has to do merely with the form of judgment, not with concepts as to their content [*Inhalt*], the distinction of infinite from negative judgments is not proper to this sciences” ([9] 9:104). Kant is cited according to the “Academy Ausgabe” (*Gesammelte Schriften*) [9] with volume and page number. The English translation is based on the Cambridge edition [10] with changes to make it more literal.

²In the third chapter of the *Dialectics*, the *Ideal*, the problem discussed is the necessity of attributing either a predicate or its negation to every thing. We shall deal only with the antinomies, which can be illustrated by the hexagon of oppositions.

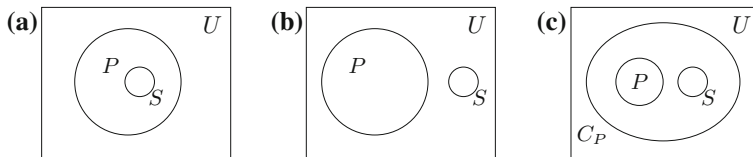


FIGURE 1. Three “qualities” of judgments according to Kant: **a** “affirmative”, the subject S is placed inside the predicate P ; **b** “negative”, the subject S is placed outside the predicate P ; **c** “infinite” (or affirmative with negative predicate), the subject S is placed outside the predicate P but within the set of candidates C_P

In Sect. 2 of this paper, we will explain infinite judgments in the context of Kant’s logic and interpret them as “qualified negations” of a predicate, i.e. negations within a predicate’s range of incompatibility. In Sect. 3 we will show that this kind of predicate negation fits the structure of the logical square and the logical hexagon, thus introducing the logical hexagon of predicate negation. In Sect. 4 we will exploit the logical hexagon of predicate negation in order to analyze Kant’s solution of the “antinomies of pure reason”. The logical hexagon allows us to visualize in a single diagram the logical structure of the antinomies and of the main dialectical argument of the *Critique of Pure Reason* in favor of transcendental idealism.

2. Infinite Judgments in Kant’s Logic

What is the difference according to Kant between denying (‘removing’) P and asserting (‘positing’) non- P ? Kant’s own explication of the difference is not very forthright, given its purported significance for the “dialectical inferences of pure reason,” and it is marred by ambiguities and even typographical errors.³ The term for judgments with negative predicates in the eighteenth century was ‘infinite’ [*unendlich*]. Many,⁴ but not most, logic textbooks of the period dealt with such judgments and others at least dealt with the negated predicates themselves under the title *termini infiniti*. Georg Friedrich Meier, whose *Auszug aus der Vernunftlehre* Kant used as the text for his logic lectures, called any judgment with a negation in its subject or predicate, as long as the copula is not negated, an infinite judgment (*judicium infinitum*).⁵ The meaning of the term *infinitus* in the logical tradition was closer to ‘indefinite’

³In both edition of the *Critique of Pure Reason* when formulating an example of infinite judgment Kant prints: the soul is *not mortal* instead of *non-mortal* or *immortal* (*nicht sterblich* instead of *nichtsterblich*). The Academy edition text and most editors amend the text so that it marks the difference. On infinite judgment in general and in relation to the antinomies, see [12], 64–80, and [8].

⁴According to Tonelli [23] 16 of the 49 eighteenth-century German logic books that he examined list infinite judgment as a separate category.

⁵[13], 82; [9] 16:636.

and it was often so translated. Kant himself tends to take the term *infinite* literally and is concerned only with negations in the predicate term.

In order better to cope with Kant's original terms, it is advisable to say some words about Kant's notion of logic. Kant conceives of categorical judgment not so much as the ascription of a property to a thing or even the placing a thing in the extension of a concept, but primarily as a relation between *concepts*. (Judgments can also connect *judgments*, as in hypothetical or disjunctive judgments.) Concepts have a *content* (*Inhalt*) and an *extent* or *sphere* (*Umfang*, *Sphäre*). Although Kant does not seem to be entirely consistent, both the content and the extent of a concept contain as a rule only concepts, not objects. There is much disagreement about the meaning of these terms in the literature; we will adopt a fairly intensionalistic reading. The 'extent' of a concept (as a genus) consists of all the concepts (as species) contained [*enthalten*] *under* it, and the 'content' of a concept (as a species) consists of all the genera contained [*enthalten*] *in* it. *Content* and *extent* are complementary terms: the 'greater' the extent, the 'smaller' the content and vice versa: "The more a concept contains *under* itself, namely, the less it contains *in* itself, and conversely."⁶ Thus *metal* (as a genus) contains *gold* (and other species) *under* it as (part of) its extent or sphere; and *gold* as a species contains *metal* (and other genera) *in* it as parts of its content (partial concepts or *Teilbegriffe*).

If we adopt the convention of using (small) capital letters for the names of concepts, we can express Kant's position as follows: GOLD contains METAL *in* itself as (part of) its *content*; METAL contains GOLD *under* it as part of its *extent* or *sphere*. METAL can be divided up (*eingeteilt*) into GOLD and other species. GOLD can be analyzed or dissected (*zerlegt*) into METAL and other partial concepts (*Teilbegriffe*), which are contained *in* the *content* of GOLD. METAL is also said to be a *cognitive ground* or *marker* for GOLD insofar as it is a *partial concept* of GOLD, which is one of the species contained *under* the genus METAL.⁷

"Every concept, as *partial concept*, is contained *in* the representation of things; [insofar] as [it is a] *ground of cognition*, i.e., as *mark* [of the representation of things], these things are contained *under* it. In the former respect every concept has a *content*, in the other an *extent*. The content and extent of a concept stand in inverse relation to one another. The more a

⁶[9] 9:95 (*Jäsche Logic*). Taken literally this is absurd: Since there is a highest genus but no lowest species, the *extent* of every genus is in a sense infinite, and the *content* of every given species is finite (at least as a *quantum discretum*). There is thus no real inverse proportionality between content and extent. However in terms of actually given species and genera a sort of proportionality may be taken to hold.

⁷"Ein jeder Begriff, als Theilbegriff, ist in der Vorstellung der Dinge enthalten, als Erkenntnisgrund, d. i. als Merkmal sind diese Dinge unter ihm enthalten. In der erstern Rücksicht hat jeder Begriff einen Inhalt, in der andern einen Umfang. Inhalt und Umfang eines Begriffes stehen gegen einander in umgekehrte Verhältnisse. Je mehr nämlich ein Begriff unter sich enthält, desto weniger enthält er in sich und umgekehrt." ([9] 9:96, *Jäsche Logic*).

concept contains *under* itself, namely, the less it contains *in* itself, and conversely.”

It would also seem that GOLD like any species can be used to divide up (*einteilen*) an appropriate genus like METAL into two exhaustive or complementary species GOLD and NON-GOLD; or BODY can be divided up into METAL and NON-METAL. Similarly when Kant says that the soul is non-mortal, we should understand: SOUL is contained *under* CREATURE (*Wesen*⁸), and CREATURE contains under it (= can be divided up [*eingeteilt*] into) MORTAL and NON-MORTAL (*tertium non datur*). Although MORTAL specifies [LIVING] CREATURE in an affirmative manner, NON-MORTAL does not specify anything positive that is not already specified in CREATURE. Thus while the affirmative judgment “the soul is mortal” places SOUL (which is contained *under* CREATURE) in the sphere of MORTAL (also contained under CREATURE), the infinite judgment “the soul is non-mortal” places SOUL in the “unlimited” *extent* of what remains when MORTAL has been excluded from CREATURE. Apparently MORTAL is determinate and NON-MORTAL indeterminate (since technically both are infinite in the number of species contained under them).

“But the infinite sphere of everything possible is thereby limited only insofar as the mortal is separated from it, and the soul is placed in the remaining space of its extent.” (A72/B97)⁹

“The extent or *sphere* of a concept is the greater, the more things stand under it and can be thought through it. . . . Now, the more things can be represented through a concept, the greater is its sphere. Thus the concept *body*, for example, has a greater extent than the concept *metal*.” ([9] 9:96)

This short account of Kantian logic straightforwardly leads us to a better understanding of “infinite judgments”. ‘The soul is non-mortal’ and ‘some men are non-learned’ are Kant’s standard (and only) paradigms of infinite judgments. Both have the peculiarity that it is more or less evident that the subject “contains” *in* its content a genus exhaustively divided up by the predicate and its negation. For instance, the soul (*anima*) is of course alive (animated); it also seems clear that humans are educable and thus are either learned or non-learned, just as the soul is either mortal or non-mortal. We do not say that a stone is non-mortal or non-learned because it does not belong to the proper genus or, in Strawson’s terms, to the *range of incompatibility* of the subject.¹⁰

⁸We read the German *Wesen* (essence, being, nature, realm) in this context as meaning (living) being or creature. In other contexts, it has other meanings.

⁹The B-version actually reads “extent of its space”.

¹⁰[20], 6. Kant already stumbled upon a similar problem in his “Versuch den Begriff der negativen Grössen in die Weltweisheit einzuführen” from 1763. While on scales open to both sides (ranging from negative to positive infinity) predicating the degree zero implies that the property belongs to the subject, predicating the same value on one-sided scales (ranging from 0 to infinity) acquires a double meaning, for not only a specific positive degree of the property is negated of the subject, but also the property as such. If e.g. we think of motion as a vector quantity (in one dimension, for the sake of simplicity), predicating the degree of

While, depending on the concept, it may be analytic that a subject falls under the predicate or its negation—as souls are either mortal or non-mortal—it is nonetheless possible that the genus exhaustively divided by the predicate and its internal negation does not actually belong to the content of the subject. In explaining the structure of the antinomies of reason Kant takes the example of a thing that smells: the genus of things that have a smell can be divided into two complementary species: good smelling and non-good smelling. However, having a smell at all is a contingent, empirical, synthetic property. It does not belong to the content of the concept of body.

“If someone said that every body smells either good or it smells not good, then there occurs a third case, namely that it does not smell (diffuse) at all, and thus both conflicting propositions can be false. If I say it [the body] is either good-smelling or it is not good-smelling (*vel [est] suaveolens vel non [est] suaveolens*), then both judgments are contradictorily opposed, and only the first is false, but its contradictory opposite, namely that some bodies are not good-smelling, includes also the bodies that *do not smell at all*.” (B531)

So far we have remained as close as possible to Kant’s wording. In order however to understand how Kant, in the resolution of the antinomies, exploits the structure of negative predicates, it is helpful to draw on a slightly formalized account. Strobach [22] has shown that the negation of predicates “un-” (or “non-”, “im-”, “-less” etc.) can be considered as an operator sharing the properties of the classical negation operator in formal logic—in particular the *tertium non datur* and the law of double negation (in two versions)—if only it is understood as a “qualified negation”, as Strobach puts it, i.e. as confined to a “set of candidates” C_P for P within which it works ([22], 96; cf. also [11], 291–295):

Def.: x is un- P iff (i) $\text{non}(x \text{ is } P)$ and (ii) $x \in C_P$.

Then the following hold as logical laws:

$$\forall x(x \in C_P \Rightarrow (Px \vee \text{un-}Px)) \quad (1)$$

$$\forall x(x \in C_P \Rightarrow (\neg \text{un-}Px \Leftrightarrow Px)) \quad (2)$$

$$\forall x(x \in C_P \Rightarrow (\text{un-un-}Px \Leftrightarrow Px)). \quad (3)$$

Footnote 10 continued

velocity “zero” implies a *state of motion*, whereas the same degree of velocity conceived as a scalar quantity is not a degree of motion but the privation [*defectus, absentia*] of the same. The inclusion of the zero-degree as one among other grades on the scale went hand in hand with a conceptual innovation: the dichotomic motion-rest was replaced by the continuous scale of the “state of motion” including “rest” as one degree of motion among others. “Rest” works on the continuous scale like a negative predicate, since it denies a positive or negative degree of motion but still implies a state of motion. In this reform dichotomies were not forsaken: motion to the right vs motion to the left were present as a dichotomy and the “state of motion” was confronted with “acceleration”.

It however does *not* hold as a logical law that

$$\neg \text{un-}Px \Leftrightarrow Px. \quad (4)$$

“Mortal” and “immortal” indeed work as contradictory terms once they are confined to the set of living things. Understood as subsets of the universal set U , however, “mortal” and “immortal” no longer bear these properties. First, there is a non-excluded middle, for non-living things are neither mortal nor immortal (living things thus are the ‘candidates’ for the predicates “mortal” and “immortal”); and secondly, as a consequence of this, the law of double negation no longer holds, for “not immortal” is verified by mortal things as well as by non-living (and hence not mortal) things. “S is not immortal” does not allow us to conclude that S is mortal. The same holds for “good smelling” and “non-good smelling”.

As we already mentioned above, Kant’s infinite judgment and its formal reconstruction as qualified negation within a set of candidates converges with Strawson’s notion of the “range of incompatibility”. “When we apply a predicate to something,” Strawson explained, “we implicitly exclude from application to that thing the predicates which lie outside the boundaries of the predicate we apply, but in the same incompatibility-range.” “‘Under six foot tall’ is incompatible with ‘over six foot tall’; but neither is incompatible with ‘aggressive’. The last expression has a different incompatibility-range from the other two.” ([20], 6, and [21], 74) Strawson also explains that negative prefixes to predicates work within the predicate’s range of incompatibility. They “do not point more emphatically to differences than to likenesses; they rather serve to underline the fact that the two are complementary.” ([20], 8) Engelbretsen, in his comment on Strawson, gives an example which makes it clear that we are dealing with the same category as Kant’s infinite judgments: “The negation of a general term, say ‘red’, can be thought of as the disjunction of all the terms in its incompatibility group (‘blue’, ‘green’, ‘pink’, etc.) So the negation of ‘red’ (‘nonred’) is ‘blue or green or pink or . . .’.” ([5], 81)

Let us sum up the essence of Kant’s theory of three qualities of judgment so far as we have analyzed it. In an affirmative judgment the subject is posited (*setzen, ponere*) in the predicate; in a negative judgment it is removed (*aufheben, tollere*) from the predicate; in an infinite judgment it is posited in a negative predicate. In the *Jäsche Logic* Kant says: “In the *affirmative* judgment the subject is thought *under* the sphere of a predicate, in the *negative* it is posited *outside* the sphere of the latter, and in the *infinite* it is posited in the sphere of a concept that lies outside the sphere of another.” (AA 9:103/4) This means that in the affirmative judgment the subject concept is said to be a species of the predicate concept; in the negative it is said not to be a species of the predicate; and in the infinite it is said to be a species of a different predicate (i.e. of the complement of the predicate). The “sphere of a concept which lies outside the sphere of an other” is what we interpret (in agreement with Strobach and Strawson) as the ‘set of candidates’ for the predicate or as the predicates’ ‘incompatibility range’. It might also be thought of as a more

general concept, such as “colored” for the negation of the predicate “blue”. “S is non-blue” would then mean that S has a color different from blue.¹¹

The solution to the problem with the negated predicate lies in the analysis of the subject term. If the subject term contains the genus of the predicate term, then the predicate and its negation exhaust the range of incompatibility. For instance, if analysis reveals that human beings are educable, then they are learned or unlearned. If a visible surface has to have color, then it must be green or non-green.

3. The Hexagon of Inner Negation

As Horn and Westerståhl ([7], 16, and [24]) have reminded us, predicate negation fits the structure of the logical square—which is not surprising, given the fact that predicate negation results in a contrary opposition.¹² In Fig. 2 we show the square of predicate negation together with the Aristotelian square of oppositions (i.e. where the universal propositions carry an existential presupposition), the logical square of the universal quantifier and the square of modal propositions. One can observe from this comparison that classical negation and predicate negation stand in direct analogy to ‘outer’ and ‘inner’ negation with respect to the universal quantifier and modal operator respectively. For this reason, we may refer to predicate negation simply as ‘inner negation’ and to the standard negation operator as ‘outer negation’. There seems indeed to be a deeper reason for this formal analogy. If we look on the four categorical propositions A, I, E, O through the eyes of term logic (which we find useful for this purpose but need not defend as an account of quantified propositions), i.e. if we take the universal affirmative proposition A “Every S is P” to be of the same form as “S is P” since both say that “is P” holds for some thing or things, then the universal negative proposition E “No S is P” is to be understood as saying that “are non-P” is true of the S so that A and E come out as contrary statements ([19], 50).

¹¹In the *Critique of Pure Reason* (B97) and also in the *Lectures on Logic* ([9] 9:103/4) this reference to a second sphere, embracing the sphere of the predicate, is less explicit and hence open for different interpretations. Kant there only asserts that infinite judgments place the subject within the “whole domain of possible beings” outside the predicate. But if it is not judgments with negative predicates within a genus that he has in mind here, the only alternative interpretation would be to envision a sort of negation with existential import. This indeed is how C.S. Peirce understood the infinite judgment ([15], 2.381, 2.326 and 4.44). This interpretation however does not explain the use of the logic of negative predicates in the solution of the antinomies (cf. Sect. 4).

¹²Inner negation or predicate negation thus can be qualified as a *paracomplete* negation which exactly means that it results in contrary oppositions. Let $v(p) \in 0, 1$ be the truth value of proposition p and \sim the operator of predicate negation, transforming “S is P” in “S is un-P”. Then \sim is characterized by the fact that $v(p) = 1 \Rightarrow v(\sim p) = 0$. From this it immediately follows that predicate negation, when applied to negated propositions $\neg p$ (with the standard negation operator \neg), results in a subcontrary opposition, as required by the logical square, for if $v(\neg p) = 0$, then $v(p) = 1$, then $v(\sim p) = 0$, then $v(\neg \sim p) = 1$, and then $v(\sim \neg p) = 1$, whereas the converse does not hold as a logical law. On paracomplete negation see [3, 18].

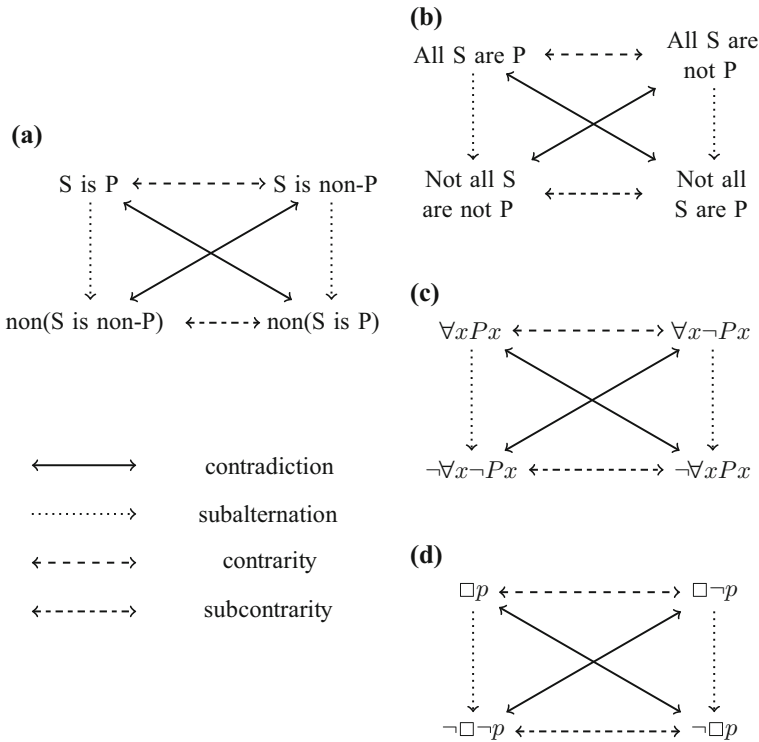


FIGURE 2. Logical squares: **a** the logical square of inner and outer negation, **b** the classical logical square of quantified propositions with existential presupposition, **c** the logical square of the universal quantifier and **d** the logical square of modal propositions

It is known (cf. [4,14]) that the logical square can be generalized to a ‘logical hexagon’ (Fig. 3), and this of course also applies to the square of inner negation (Fig. 4). The logical hexagon, to recall its basic features, is an interesting and useful generalization of the square of oppositions. Technically speaking, the basic idea consists in adding two additional corners, one above the upper edge, connecting the A- and the E-corners, which is defined as the disjunction of these two corners $A \vee E$ and happened to be called the U-corner; and the second one, the Y-corner, beneath the lower edge I to O, defined as their conjunction $I \wedge O$. Adding these two corners results in a hexagonal structure. As both A and E imply $A \vee E$, the U -corner stands in a relation of subalternation to A and E . The same applies to I and O which are implied by Y . Furthermore, the new corners are contradictories of each other, as are A and I and E and O , such that each pair of opposite corners of the hexagon is constituted by a contradiction. In addition, the new corners also fit well into the whole of contrary and subcontrary relations. Indeed, the hexagon can be conceived as a 6-pointed star resulting from the superposition of the

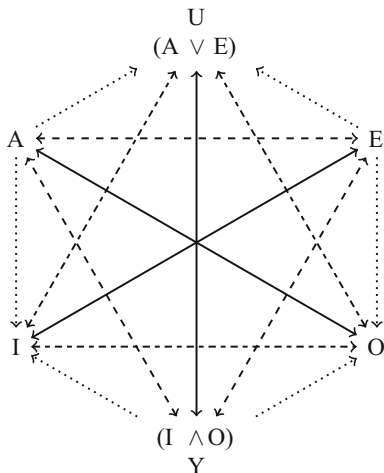


FIGURE 3. The logical hexagon

two triangles of contrarities $A-E-Y$ and subcontrarities $I-O-U$. Whereas this generalization of the square is straightforward and technically rather simple (indeed there are further, sophisticated generalizations), it derives its interest from the meaning which can be attributed to the additional two corners. In the hexagon of quantified propositions, the Y -corner represents according to its definition the proposition “Some, but not all, S are P ”, which fits quite well the meaning of “some” in ordinary discourse. In the hexagon of modal propositions, the Y -corner—“Neither non- p nor p is impossible” or “it is possible but not necessary that p ”—obviously corresponds to the contingency of p . Finally, in the deontic hexagon, Y represents “it is allowed to”. Here we already stumble over an interesting link to our question of predicate-negation, since the Y -corner obviously captures what one often wants to say in non-technical discourse by negating the A -corner. “Not all” often means “some but not all”, and “not necessarily” “possibly but not necessarily” etc. The relevant but implicit opposition referred to in this kind of negations is the contrary opposition between A and Y , which shares the formal structure of our qualified negation (“not blue” meaning that the object has a colour, but that it is not blue). The logical hexagon proves helpful as a tool of conceptual analysis with a surprisingly wide range of applications. Béziau has already applied it to the Kantian notions of analytic, synthetic, a priori and a posteriori propositions ([4], 12, reproduced here as Fig. 5). We will see below that it also extremely revealing to apply the hexagon to Kant’s solution of the antinomies of pure reason, exploiting the structure of predicate negation.

When we apply the logical hexagon to predicate negation, we can make an observation that will be of utmost importance later on, namely that the Y - and U -corners acquire an intuitive meaning (Fig. 4). On the U -corner we have the proposition that S is either P or un- P . While this statement seems not informative on the first look, it reveals important information about S on

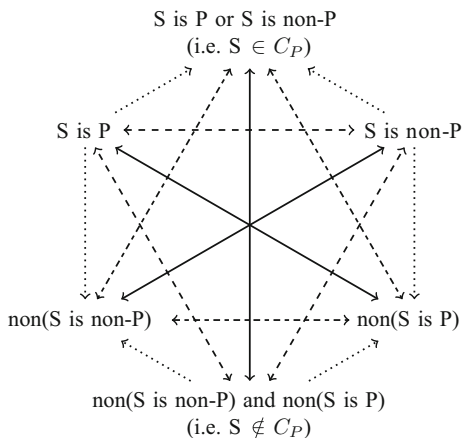


FIGURE 4. The logical hexagon of inner and outer negation

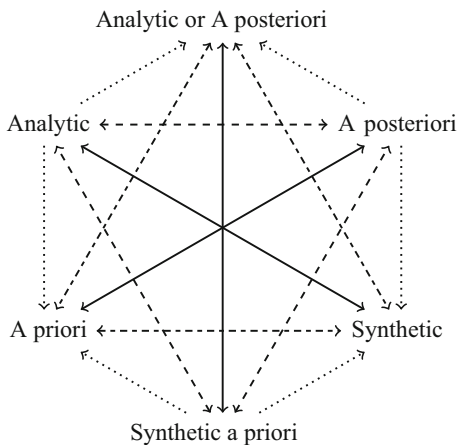


FIGURE 5. The “Kantian hexagon” from [4], 27

a more close investigation. Indeed U does not tell us whether or not S is P , but it does tell us that S is a candidate for P : $S \in C_P$. That S is either mortal or immortal implies that S is a living thing. For the opposite Y -corner, the opposite holds. That S is neither P nor un- P implies that S does not count among the candidates for P . That S is neither mortal nor immortal means that S is not alive and hence will neither die nor live eternally.

4. Inner Negation and Kant's Solution of the Antinomies of Pure Reason

Let us now turn to the antinomies of pure reason. If one keeps in mind the formal properties of negated predicates (i.e. qualified negations of predicates),

the solution of the antinomies is straightforward. According to Kant the power of cognition (understanding) encounters a fourfold antinomy of reason when trying to answer basic questions of cosmology. The four problems that lead to antinomies are ordered according to the architectonics given by the categories: Each of the four classes of categories (quantity, quality, relation, modality) has its own antinomy ordered in one pair of mathematical antinomies and a second pair of dynamical antinomies. The positions confronted in the antinomies are on the whole those taken by early eighteenth-century empiricist and rationalist metaphysics: the “thesis” position is very close to that taken by Samuel Clarke (and Isaac Newton) in the Leibniz-Clarke Correspondence [17]; and Leibniz certainly provides much of the material for the antitheses.¹³

All four pairs of claims can be formulated such that one of the propositions uses a particular predicate while the other uses the negation of that predicate (Table 2):¹⁴

For every claim, Kant also presented a proof; these proofs are more or less convincing, as the case may be. However, what poses a real challenge for the interpretation of Kant’s text is the fact that Kant himself quite explicitly endorsed these proofs, while at the same time promising to overcome the conflicting standpoints. Let us have a closer look on the first antinomy, “the world is finite” vs “the world is infinite”. Key to understanding Kant’s reasoning is to observe that the proofs of both the thesis and the antithesis are *indirect* or *apagogical* proofs, i.e., proofs by *reductio ad absurdum*. Each proof thus consists in a first step that assumes the opposite thesis and shows its absurdity. The proofs of both theses and antitheses are thus *refutations* of their respective antagonists. Only in a second step does the argument infer the truth of the proposition to be proved from the established falsity of its opposite. This leads to the paradox that Kant seems to endorse proofs of contradictory claims.

However, if we keep in mind however that for negated predicates the law of double negation does not generally hold, the paradox immediately dissolves. Taken without restriction to the set of candidates, “S is P” and “S in non-P” are not contradictories but merely contrary claims, which can both be false at the same time. The falsity of both claims indeed is established in the refutation part of the proofs, which Kant can perfectly well still endorse after recognizing the mere *contrariety* of the propositions. The second step of the proofs—which start from the absurdity of the consequences of the assumption and then infer the truth of its contrary—is not valid since it cannot apply the law of double negation [25].

In one of his drafts for an answer to an Academy prize question on the “progress made in metaphysics since Leibniz and Wolff,” Kant himself made this point explicit. There he compares the apparent contradictions of the First

¹³[1]; [12], Ch. 2.

¹⁴We cite the shorter and clearer formulations given to the antinomies in the *Prolegomena* of 1783 ([9] 4:339). Which of the two propositions is formulated as an affirmative judgment and which as an infinite judgment can vary: Kant’s second order discussion of the form of the First Antinomy in the Zeno passage (B530–33) uses the opposition: the world is *infinite* (Antithesis) vs. the world is *non-infinite* (Thesis).

TABLE 2. The four antinomies of pure reason. In the two “mathematical antinomies” (1 and 2) thesis and antithesis are according to Kant contrary opposites, in the two “cosmological antinomies” (3 and 4) they are subcontrary opposites

	Thesis	Antithesis
First antinomy: finite vs infinite size of the world	“The world has, as to time and space, a beginning (limit).”—The world is finite in space and time	“The world is, as to time and space infinite.”—The world is infinite in space and time
Second antinomy: atoms vs plenum	“Everything in the world is constituted out of the simple.”—The number of parts of a (compounded) body is finite	“There is nothing simple, but everything is composite.”—The number of parts of a (compounded) body is infinite
Third antinomy: indetermin-ation vs determination	“There are in the world causes through freedom.”—Some events are physically undetermined	“There is no freedom but all is nature.”—All events are physically determined
Fourth antinomy: necessity vs contingency	“In the series of world causes there is some necessary being.”—Some entities are non-contingent	“There is nothing necessary in the world, but in this series all is contingent.”—All entities are contingent

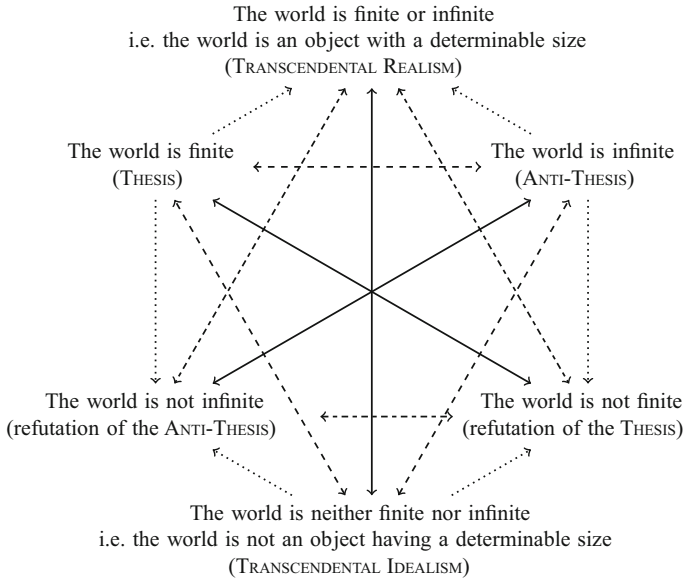


FIGURE 6. The logical hexagon of inner and outer negation applied to the first antinomy of pure reason

and Second Antinomies to *contrary* oppositions in logic and those of the Third and Fourth Antinomies to *subcontrary* oppositions. On the First Antinomy Kant writes:¹⁵

... the conflict between these propositions of reason is not merely a logical conflict of analytical opposition (*contradictorie oppositorum*), i. e., a mere contradiction, for in that case, if one of them is true, the other would have to be false, and conversely. E. g., *the world is infinite in space*, compared to the antithesis, *it is not infinite in space*. It is, rather, a transcendental conflict of synthetic opposition (*contrarie oppositorum*), e. g., *the world is finite in space*, a proposition which says more than is required for logical opposition; ... so that these two propositions can both of them be false—like two judgments in logic that are opposed to one another as contraries [*Widerspiel*] (*contrarie opposita*).

There is a last step, which completes the solution of the first antinomy. If indeed both thesis and antithesis have been proven to be false, this means that the *tertium*—which for contrary predicates is not excluded—actually holds. So what does the *tertium* consist in? Both thesis and anti-thesis tacitly presupposed that the world has a definite size, i.e. that the world as a whole is a

¹⁵ *Progress of Metaphysics*, [9] 20:291; see also 20:327–29.

possible object of experience the size of which can be measured. This position is what Kant called “Transcendental Realism”. This presupposition now turns out to be false. The subject term does not contain the genus exhausted by the predicates. If the world has no determinate size, it is neither finite nor infinite. The *tertium* which actually holds thus is the negation of the tacit presupposition—or the claim that the world as a whole is not a possible object of experience. But this amounts to the view of Transcendental Idealism, i.e., the view which Kant aimed to establish in his first *Critique*. The solution of the antinomies thus amounts to an indirect proof of Transcendental Idealism by rendering absurd its adversary, Transcendental Realism, by showing that the latter leads into the antinomies of pure reason (cf. B534 and [11], 288). The *reductiones ad absurdum* of the thesis and anti-thesis are integrated into a large scale *reductio* of Transcendental Realism. When we apply the hexagon of inner negation to the first antinomy (Fig. 6), this structure becomes immediately visible, for the additional U- and Y-corners exactly correspond to Transcendental Idealism and Transcendental Realism. This application of the logical hexagon seems extremely helpful, because it allows us to show in one single diagram both the logical structure of the solution of the antinomies (i.e. to show that the opposition is not one of contradiction but rather a merely one of contrariety) and the logical structure of Kant's project in the Dialectics of the first Critique, i.e. to refute transcendental realism (the U-corner) by rendering it absurd.

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